Math 5C Discussion Problems 3

Power Series

1. Find the radius and interval of convergence for the following power series.
   (a) $\sum 5^n x^n / n^n$
   (b) $\sum x^n / \ln n$
   (c) $\sum (-1)^{n+3} x^{2n+1} / (n!)^2$
   (d) $\sum x^n / (1 + n^2)$
   (e) $\sum x^n / (\ln n)^n$

2. Assume that $\sum a_n 3^n$ converges absolutely. What can you say about the convergence of each of the following?
   (a) $\sum a_n 4^n$
   (b) $\sum a_n 2^n$
   (c) $\sum a_n (-3)^n$
   (d) $\sum a_n$

3. Assume that $\sum a_n 4^n$ converges conditionally. What can you say about the convergence of each of the following?
   (a) $\sum a_n 5^n$
   (b) $\sum a_n 2^n$
   (c) $\sum a_n (-3)^n$
   (d) $\sum a_n$

4. Given that $f(x) = \sum a_n x^n$ converges in a neighborhood of zero, what is $f'''(0)$?

5. Which of the following are true for x near zero?
   (a) $x^2 = O(x^3)$
   (b) $x^3 = O(x^2)$
   (c) $x^2 + x^4 = O(x^3)$

6. Assume that
   $$y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$
   Find the first two nonzero terms of a series expansion of y^2 in terms of x. Write the remainder using $O(x^n)$ notation.

7. Assume that
   $$z = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots$$
   $$y = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
   Find the first two nonzero terms of a series expansion of z in terms of x. Write the remainder using $O(x^n)$ notation.

Taylor Series

1. Using the Taylor series of exp, sin and cos, show that $e^{i\theta} = \cos\theta + i\sin\theta$.

2. Prove these identities for arbitrary real numbers x.

   (a) $\sinh(ix) = i\sin x$

   (b) $\cosh(ix) = \cos x$

3. Use a Taylor series to approxmate the following.

   (a) $e^{0.3}$ with an error less than 0.1.

   (b) $\sin(0.1)$ with an error less than 10^{-10}

   (c) $\displaystyle\int_0^1 e^{-x^2}\,dx$ with 1 decimal place accuracy (error less than 0.05)

   (d) $\displaystyle\int_0^1 \frac{1-\cos x}{x}\,dx$ with 2 decimal place accuracy (error less than 0.005)

4. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin x^3}{x^3}$

   (b) $\displaystyle\lim_{n\to\infty} n(e^{1/n} - 1)$

   (c) $\displaystyle\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{\arctan(x^4)}$

   (d) $\displaystyle\lim_{x\to 0} \frac{e^{-x^2}\cos x - 1}{x^2}$

   (e) $\displaystyle\lim_{x\to 0+} \frac{x^x - 1}{x\ln x}$

   (f) $\displaystyle\lim_{x\to 0} \frac{e^{\sin x} - 1 - x}{x^2}$

   (g) $\displaystyle\lim_{n\to\infty} n^3 \left(\arctan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right)$

   (h) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n^2} e^{-n}$

5. Evaluate the following series.

   (a) $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

   (b) $\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$

   (c) $1 + \frac{1}{2\cdot 3!} + \frac{1}{4\cdot 5!} + \frac{1}{8\cdot 7!} + \cdots$

   (d) $1 + \frac{2}{3} + \frac{3}{9} + \frac{4}{27} + \frac{5}{81} + \cdots$

   (e) $1 + \frac{2^2}{7} + \frac{3^2}{7^2} + \frac{4^2}{7^3} + \frac{5^2}{7^4} + \cdots$

   (f) $1 + \frac{1}{2\cdot 7} + \frac{1}{3\cdot 7^2} + \frac{1}{4\cdot 7^3} + \frac{1}{5\cdot 7^4} + \cdots$

Fourier Series

1. For each of the following, compute $\int_{-\pi}^{\pi} f(x)\sin x\,dx$. No integrals need be computed.
   (a) $f(x) = \cos x + \cos 10x + \sin 3x$
   (b) $f(x) = \sin^2 x$
   (c) $f(x) = \cos^3 x$
   (d) $f(x) = \sin x \cos x$

2. For each of the following 2π-periodic functions, compute the Fourier series.
   (a) $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$
   (b) $f(x) = x$ when $-\pi < x < \pi$
   (c) $f(x) = x$ when $0 < x < 2\pi$
   (d) $f(x) = e^{-|x|}$ when $-\pi < x < \pi$

3. Show that the product of even functions is even, the product of odd functions is odd, and the product of an even and an odd function is odd.

4. (a) Suppose that f is an even function. Show that $b_n = 0$ for all n.
   (b) Suppose that f is an odd function. Show that $a_n = 0$ for all n.

5. Let $f(x) = x^2$ for $0 < x < 2\pi$, extended periodically.
   (a) Is f even or odd?
   (b) Show that the Fourier series of f is
   $$f(x) \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2} + \frac{\pi \sin nx}{n}\right)$$
   (c) Use the Fourier series to evaluate $\sum_{n=1}^{\infty} n^{-2}$ by setting $x = 0$.
   (d) Use the Fourier series to evaluate $\sum_{n=1}^{\infty} (-1)^n n^{-2}$.
   (e) Use Parseval's identity to evaluate $\sum_{n=1}^{\infty} n^{-4}$.

6. Use Parseval's identity to deduce the following fact. Given a 2π-periodic function f with
   $$\int_{-\pi}^{\pi} f(x)^2\,dx < \infty,$$
   it follows that
   $$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0.$$

7. From the expansion
   $$-\ln\left(2\sin\frac{x}{2}\right) = \sum_{n=1}^{\infty}\frac{\cos nx}{n}, \qquad 0 < x < 2\pi,$$
   use Parseval's identity to evaluate the integral
   $$\int_0^{\pi} \ln^2\left(2\sin\frac{x}{2}\right)\,dx.$$

Series: Miscellany

1. We know that $(1+1/n)^n \to e$ when $n \to \infty$. Using series, find the number a that satifies

$$\left(1+\frac{1}{n}\right)^n = e + \frac{a}{n} + O\left(\frac{1}{n^2}\right) \qquad \text{for large } n$$

2. Show that, for large integers n,

$$\sin(2\pi n!e) = \frac{2\pi}{n+1} + O(n^{-2})$$

   Use this to evaluate $\lim_{n\to\infty} n\sin(2\pi n!e)$.

3. Use what you know about approximating functions to determine converegence of these series.
   (a) $\sum\left(1-\cos\left(\frac{1}{n}\right)\right)$
   (b) $\sum\left(1-n\sin\left(\frac{1}{n}\right)\right)$
   (c) $\sum\log\left(n\sin\left(\frac{1}{n}\right)\right)$ (use the previous part and limit comparison)
   (d) $\sum\left(1-e^{1/n^\alpha}\right)$ (the answer depends on α)

4. Prove that for $-1 < x < 1$,

$$\left(\sum_{n=0}^{\infty} x^n\right)^2 = \sum_{n=1}^{\infty} nx^{n-1}$$

5. (Not for the faint of heart!) About 20 years ago, renowned mathematical physicist V.I. Arnold declared that modern students of mathematics are poorly trained. Sobolev embedding theorems and abelian category theory are nothing if you can't compute the average value of $\sin^{100} x$ within 10% in under five minutes (or so he claims). Arnold then posted a "mathematical trivium," that is, 97 problems any math student should be able to solve with minimal effort. Here's problem number 2 from the list: evaluate

$$\lim_{x\to 0}\frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

6. (See above problem description) For those who can't get enough Arnold, here's trivium number 18. Evaluate

$$\int\cdots\int_{\mathbb{R}^n} \exp\left(-\sum_{j=1}^{n}\sum_{k=1}^{j} x_j x_k\right) dx_1 dx_2 \cdots dx_n.$$

Harmonic Functions

1. Which of these functions are harmonic on $\mathbb{R}^2$?
   (a) $f(x, y) = e^x \sin y$
   (b) $f(x, y) = x^2 - y^2$
   (c) $f(x, y) = x^2 + y^2$
   (d) $f(x, y) = \ln(x^2 + y^2)$
2. Give an example of harmonic functions f and g so that fg is not harmonic.
3. Assume that $f(x, y)$ is harmonic on $\mathbb{R}^2$. Show that $zf(x, y)$ is harmonic on $\mathbb{R}^3$.
4. Assume the maximum principle is true. Show that the uniqueness theorem follows. Hint: $f - g$
5. Give a reason why $\nabla(\nabla \cdot \mathbf{F}) \neq \nabla \cdot (\nabla f)$.
6. Let f be harmonic on $\mathbb{R}^3$ and B be a ball. Use the divergence theorem to show that
$$\iint_{\partial B} f \nabla f \cdot d\mathbf{A} = \iiint_B \|\nabla f\|^2 \, dV.$$
7. Use the previous problem to prove the following: If f is harmonic and $f = 0$ on the surface of a ball, then $f = 0$ everywhere inside the ball.
8. Use the uniqueness for harmonic functions to prove the following. Suppose smooth scalar functions f and g on $\mathbb{R}^3$ are equal on the surface of a ball B. Suppose that $\Delta f = \Delta g$; show that $f = g$ inside B.
9. Use the fact that $\ln(x^2 + y^2)$ is harmonic in $\mathbb{R}^2$ (except at the origin) to answer the following: Find a function $f(x, y)$ which is equal to 10 on the circle $x^2 + y^2 = 1$, equal to 20 when $x^2 + y^2 = 4$, and is harmonic when $1 < x^2 + y^2 < 4$. What are the odds that someone else got the same answer?
10. Find a harmonic function $f(x, y, z)$ which is identically 1 on surface of the sphere $x^2 + y^2 + z^2 = 1$.
11. Assume that f and g are harmonic functions in $\mathbb{R}^3$ and B is a ball in $\mathbb{R}^3$. Use the divergence theorem to show that
$$\iint_{\partial B} f \nabla g \cdot d\mathbf{A} = \iint_{\partial B} g \nabla f \cdot d\mathbf{A}.$$
12. Let B be the unit ball in $\mathbb{R}^3$ centered at $(10, 10, 10)$ and define $f(x, y, z) = z \ln(x^2 + y^2)$. Evaluate
$$\iint_{\partial B} f \, d\sigma.$$
13. Let B be the unit ball in $\mathbb{R}^3$ centered at $(1, 2, 2)$ and define $f(x, y, z) = x \ln(y^2 + z^2)$. Explaining your reasoning, evaluate
$$\iint_{\partial B} f \, d\sigma.$$